Journal of Statistical Physics, Vol. 112, Nos. 1/2, July 2003 (© 2003)

Uniform Cantor Singular Continuous Spectrum for Nonprimitive Schrödinger Operators

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Received July 15, 2002; accepted February 20, 2003

It is shown that some Schrödinger operators, with nonprimitive substitution potentials, have pure singular continuous Cantor spectrum with null Lebesgue measure for all elements in the respective hulls.

KEY WORDS: Nonprimitive substitution; singular continuous spectra; uniform spectrum.

1. INTRODUCTION

It is interesting that zero (Lebesgue) measure Cantor and pure singular continuous spectrum have been the rules for finite valued almost periodic (discrete) Schrödinger operators in one dimension.[1–4, 9, 16, 20, 21, 23, 24] Virtually all rigorous results have pointed to that direction, including important cases of primitive substitutions and Sturmian potentials (see the review of ref. 4 and references therein). A latent exception for singular continuous spectrum is the Rudin–Shapiro substitution for which numerical simulations indicate the occurrence of a point component in the spectrum,[12, 13] although it is still an open mathematical problem.

Such potentials give rise to strictly ergodic (i.e., uniquely ergodic and minimal) dynamical systems (Ω, T), where Ω is the hull of the given potential in $\mathbb{Z}$ constructed from the left shift operator T (see ahead for more details). In terms of the elements in the hull, there are in the literature three kinds of the mentioned spectral results which, with increasing degree of generality, can be stated as *generic* (valid in a dense G_δ set), *full measure* (valid in a set of total invariant measure) and *uniform* (valid for all elements

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0022-4715/03/0700-0357/0

in the hull). It is worth bringing up that due to minimality the spectrum, as a set, does not depend on the element in the hull, so that if the zero Lebesgue measure is verified for some potential, it then holds for all of them. Notice that the zero Lebesgue measure property has lately gotten an attractive discussion in ref. 20.

Clearly the uniform results are the most rare and, in the aperiodic setting, have been gotten just for Sturmian,[7] quasi-Sturmian,[6] and the Period Doubling substitution[5] potentials. The general strategy is to analyze the possibility for Gordon-type arguments[8, 24] for all elements in the hull (taking into account the almost periodicity, via partitions), excluding the point spectrum (see ahead).

Recently, we have studied a rather broad class of nonprimitive substitutions[11] (we call it "ζ-class") and have gotten cases of aperiodic sequences with pure singular continuous spectrum for the corresponding Schrödinger operators, for potentials in generic and/or full measure sets in the hull. In spite of lacking of Perron–Frobenius theorem, also for the nonprimitive cases studied it was possible to prove strictly ergodicity, a very important ingredient in such considerations. Nevertheless, nothing was said about the Lebesgue measure of the spectrum and uniform results. It is the aim of this work to fill in this gap, with examples of uniform results, including the original nonprimitive substitution we have studied in ref. 10.

This work is organized as follows. In the second section we give basic definitions, present the subclasses of the ζ-class for which we have obtained uniform results and enunciate the main outcomes of this work (Theorems 1 and 2); in Section 3 we discuss some general facts about the ζ-class substitutions and, as a simple application of results in ref. 20, present the proof of Theorem 1; in Section 4, following the strategy of partitions of refs. 5 and 7, and a link to the Period Doubling substitution, we proceed to the proof of our uniform spectral results (Theorem 2).

2. MAIN RESULTS

Our discussion will be restricted to substitution potentials assuming just two values, so we consider an alphabet $\mathscr{A} = \{a, b\}$. As usual, $\mathscr{A}^*$ denotes the set of all words of finite length and $\mathscr{A}^{\mathbb{N}}$ the set of all right infinite words with letters in $\mathscr{A}$. A *substitution* is a map $\xi\colon \mathscr{A} \to \mathscr{A}^*$. ξ can be extended homomorphically to $\mathscr{A}^*$ and $\mathscr{A}^{\mathbb{N}}$ by concatenation, for instance, $\xi(abaa) = \xi(a)\,\xi(b)\,\xi(a)\,\xi(a)$ and $\xi(aba\ldots) = \xi(a)\,\xi(b)\,\xi(a)\ldots$. A *substitution sequence* is a fixed point $\bar{u}$ of ξ in $\mathscr{A}^{\mathbb{N}}$, i.e., $\xi(\bar{u}) = \bar{u}$. The existence of a such fixed point is ensured by the following conditions (see ref. 22, Proposition V.1): there is a letter a in $\mathscr{A}$ such that $\xi(a)$ begins with a and the length of $\xi^k(a)$ goes to infinity as $k \mapsto \infty$ (ξ^k denotes the kth iterate of ξ;

it is supposed that $\xi^0(a)=a$). Given P and Q in $\mathscr{A}^*$, $\#_P Q$ denotes the number of occurrences of P in Q and $|P|$ the length of P. A substitution rule ξ is called *primitive* if there is $j\in\mathbb{N}$ such that $\#_c\xi^j(d)\geqslant 1$, for every $c, d\in\mathscr{A}$.

We recall the "ζ-class" of nonprimitive substitutions (introduced in refs. 10 and 11):

$$\zeta(a)=\underbrace{a\ldots a}_{A_1}\underbrace{b\ldots b}_{B_1}\underbrace{a\ldots a}_{A_2}\ldots\underbrace{b\ldots b}_{B_N}\underbrace{a\ldots a}_{A_{N+1}},\qquad \zeta(b)=b, \tag{1}$$

with A_j, $B_j\geqslant 1$, $j=1,\ldots, N$, and $A_{N+1}\geqslant 1$, being the number of letters in each block. ζ is clearly nonprimitive since $\#_a\zeta^j(b)=0$, $\forall j$.

Consider on $\mathscr{A}^{\mathbb{N}}$ ($\mathscr{A}^{\mathbb{Z}}$) the point convergence topology generated by the metric

$$d(u, v)=\sum_n\frac{|u_n-v_n|}{2^{|n|}},\qquad u=(u_n),\quad v=(v_n),$$

with $n\in\mathbb{N}$ ($n\in\mathbb{Z}$). Given $\bar{\eta}$ a substitution sequence associated to ζ in (1), consider the periodic sequences in $\mathscr{A}^{\mathbb{Z}}$

$$\eta_n=\ldots\zeta^n(a)\,\zeta^n(a)\cdot\zeta^n(a)\,\zeta^n(a)\ldots, \tag{2}$$

with the dot indicating the position of the zero index term. Since $\zeta(a)$ begins and ends with a, (η_n) is a Cauchy sequence and one gets a well-defined limit

$$\eta=\lim_{n\to\infty}\eta_n$$

in $\mathscr{A}^{\mathbb{Z}}$, called the bilateral substitution sequence generated by ζ (if $A_j=1$ for all j, some adaptation is needed in order to guarantee the almost periodicity of η; see Section 4.3 for a particular occurrence).

Let T: $\mathscr{A}^{\mathbb{Z}}\mapsto\mathscr{A}^{\mathbb{Z}}$ be the left shift $(Tx)_n=x_{n+1}$; the *hull* of x in $\mathscr{A}^{\mathbb{Z}}$ is defined as

$$\Omega=\Omega(x)=\text{Closure of }\{T^n x\text{: } n\in\mathbb{Z}\}\text{ in }\mathscr{A}^{\mathbb{Z}}.$$

We notice that for a recurrent sequence x its hull Ω is a compact and T-invariant subset of $\mathscr{A}^{\mathbb{Z}}$ with $T\Omega(x)=\Omega(x)$ (ref. 22, Chapter V), so that the dynamical system $(\Omega(x), T)$ is well defined.

From now on we always denote by $\eta=\eta_\zeta$ the bilateral substitution sequence generated by ζ and $\Omega_\zeta=\Omega(\eta)$ its hull.

Given an injective real function $f: \mathscr{A} \mapsto \mathbb{R}$, we associate a sequence in $l^\infty(\mathbb{Z})$ to each $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega$ by $(f(\omega_n))_{n \in \mathbb{Z}}$, which we again denote by ω and call it a *substitution potential*. Therefore, every point $\omega \in \Omega$ defines a bounded self-adjoint operator H_ω on $l^2(\mathbb{Z})$ by

$$(H_\omega u)_n = u_{n+1} + u_{n-1} + \omega_n u_n.$$

As usual, we shall indicate the spectrum of a self-adjoint operator H by $\sigma(H)$ and by $\sigma_{ac}(H)$ its absolutely continuous spectrum.

Now we state the main results of this paper, and postpone their proofs to later sections:

Theorem 1. Given an aperiodic substitution in the ζ-class (1), the spectrum of the associated Schrödinger operator H_ω is a Cantor set with null Lebesgue measure, for every $\omega \in \Omega_\zeta$.

Consider the following particular substitutions in the ζ-class:

$$\begin{aligned} \zeta_1(a) &= a\underbrace{b\ldots b}_{B_1}aa\underbrace{b\ldots b}_{B_1}a \\ \zeta_2(a) &= aa\underbrace{b\ldots b}_{B_1}aa \\ \zeta_3(a) &= a\underbrace{b\ldots b}_{B_1}a\underbrace{b\ldots b}_{B_2}a\underbrace{b\ldots b}_{B_1}a, \qquad B_1 \neq B_2. \end{aligned} \tag{3}$$

Theorem 2. For each substitution in (3) the corresponding Schrödinger operators H_ω have pure Cantor singular continuous spectrum with null Lebesgue measure for all ω in the respective hulls.

3. NULL LEBESGUE MEASURE

Before proceeding to the proof of Theorems 1 and 2 we gather and amend some general properties of the ζ-class substitution. The suitable properties of ergodicity and aperiodicity that we mentioned in the Introduction are sufficient to exclude absolutely continuous spectrum for all elements in the hull. This is gotten by combining the results of Kotani:[17] for aperiodic ergodic potentials taking only a finite number of values, the set of potentials with no absolutely continuous spectrum has full ergodic measure; Last and Simon:[18] for minimal subshift potentials the absolutely continuous spectrum is ω-independent in the hull. Thus to exclude absolutely continuous spectrum we have to give conditions for a substitution in the ζ-class to be aperiodic (in this case we shall also say that ζ is aperiodic) and its associated hull be minimal and ergodic. These conditions are written out in the following propositions, which complement a condition in ref. 11.

Proposition 1. If either $A_j \geqslant 2$ for some $1 \leqslant j \leqslant (N+1)$ or the B_j's are not all equal (or both), then the resulting substitution sequence η is not ultimately periodic.

Proof. For the case $A_j \geqslant 2$ see Lemma 1 in ref. 11. In the other case ζ takes the form (with $N \geqslant 2$)

$$\zeta(a) = a\underbrace{b\dots b}_{B_1}a\underbrace{b\dots b}_{B_2}\dots a\underbrace{b\dots b}_{B_N}a.$$

Due to the definition of η, we restrict the proof to strictly positive index of η, i.e., to $\bar{\eta}$. We consider first the case of periodic $\bar{\eta}$ and then reduce the case of ultimately periodic sequence to an argument of the periodic one.

Suppose that $\bar{\eta}$ is periodic and denote the first minimal period block of $\bar{\eta}$, by P ($|P| \geqslant 2$). By the very definition of ζ the minimal period block ends in the last b of a block B_j, i.e., $\eta_{|P|} = b$ and $\eta_{|P|+1} = a$. Let n_c be the unique integer such that

$$|\zeta^{n_c - 1}(a)| \leqslant |P| < |\zeta^{n_c}(a)|.$$

The first entries of $\bar{\eta}$ are

$$\bar{\eta} = \overbrace{\underbrace{\zeta^{n_c-1}(a)\,\underbrace{b\dots b}_{B_1}\dots\zeta^{n_c-1}(a)\,\underbrace{b\dots b}_{B_j}}_{P}\dots\underbrace{b\dots b}_{B_N}\zeta^{n_c-1}(a)}^{\zeta^{n_c}(a)}\,\underbrace{b\dots b}_{B_1}\zeta^{n_c}(a)\,\underbrace{b\dots b}_{B_2}\zeta^{n_c}(a)\dots .$$

The choice of n_c entails that a period block P starts at position $(|\zeta^{n_c}(a)| + B_1)$. Similarly, it can be seen that $\bar{\eta}$ is periodic with period $|\zeta^{n_c}(a)| + B_j$, for all $j = 1,\dots, N$. As all periods must be integer multiples of $|P|$, this implies that all B_j must be equal (here, we use that $|P| > B_j$). Therefore $\bar{\eta}$ (and consequently η) is not periodic.

Suppose now that $\bar{\eta}$ is ultimately periodic, so that there is an integer $k \geqslant 1$ such that $\bar{\eta}$ is periodic, with period τ, after its kth position ($k \geqslant 2$). Choose an integer m such that $|\zeta^m(a)| > k$ and $|\zeta^m(a)| > \tau$. The first entries of $\bar{\eta}$ are

$$\zeta^{m+1}(a)\ \underbrace{b\dots b}_{B_1}[\zeta^{m+1}(a)]\ \underbrace{b\dots b}_{B_2}\dots$$

and writing out the above second block $\zeta^{m+1}(a)$ one gets, for the beginning of $\bar{\eta}$,

$$\zeta^{m+1}(a)\ \underbrace{b\ldots b}_{B_1}[\zeta^m(a)\ \underbrace{b\ldots b}_{B_1}\ldots\zeta^m(a)\ \underbrace{b\ldots b}_{B_N}\zeta^m(a)]\ \underbrace{b\ldots b}_{B_2}\ldots.$$

The choice of m implies that a period block starts at position $(|\zeta^{m+1}(a)|+B_1)$ and from it, to each $(|\zeta^m(a)|+B_j)$-block. As stated before this implies that B_j are the same for all $1\leqslant j\leqslant N$. This contradiction shows that $\bar{\eta}$ and η are not ultimately periodic. ∎

Proposition 2. The subshift dynamical system (Ω_ζ, T) associated with a substitution in the ζ-class (1) is strictly ergodic.

Proof. The periodic case is well known; for the aperiodic one see Proposition 1 in ref. 11. ∎

In the primitive substitution case this fact is a consequence of the important Perron–Frobenius Theorem (for details see ref. 22, Sections V.3–V.5), which does not apply to our case and a specific proof of strictly ergodicity was necessary (see also the remark at the end of this section for another proof than the one referred to above).

The above propositions are important to conclude

Proposition 3. If ζ is aperiodic (e.g., as in Proposition 1), then $\sigma_{ac}(H_\omega)=\emptyset$, for all $\omega\in\Omega_\zeta$.

Proof. See Proposition 2 in ref. 11. ∎

Due to results of ref. 20, in order to establish Cantor spectrum with null Lebesgue measure for the one-dimensional Hamiltonian with aperiodic ζ-class potentials, it is sufficient to verify that these substitutions are linearly repetitive, i.e., for each of them there exists a $C>0$ such that for every $n\in\mathbb{N}$, every finite word (also called factor) of η of length n is a word of every factor of length Cn.

Proposition 4. Every substitution in the ζ-class is linearly repetitive.

Proof. Given ζ as in (1), set $f=\sum_{i=1}^{N+1}A_i$, $h=\sum_{i=1}^{N}B_i$, and $R_j=(f^j-1)/(f-1)$, so that[11]

$$|\zeta^j(a)|=f^j+hR_j.$$

Therefore, there exist positive numbers s, r such that

$$sf^j\leqslant|\zeta^j(a)|\leqslant rf^j,\qquad\forall j\in\mathbb{N}.$$

By almost periodicity (which follows from minimality) there exists $J > 0$ for which every factor

$$\eta_{k+1}\eta_{k+2}\cdots\eta_{k+J}$$

of η of length J contains $\zeta^2(a)$. Now, given a positive integer n, take j_n such that

$$|\zeta^{j_n-1}(a)| < n \leqslant |\zeta^{j_n}(a)|.$$

In this way, any factor of the form

$$w = \zeta^{j_n}(\eta_{k+1})\,\zeta^{j_n}(\eta_{k+2})\cdots\zeta^{j_n}(\eta_{k+J})$$

contains $\zeta^{j_n+2}(a)$ and hence all words of length n of η. Since

$$|w| \leqslant |\zeta^{j_n}(a)| \times J \leqslant \frac{r}{s}\, s f^{j_n-1} f J \leqslant \frac{r}{s}\, |\zeta^{j_n-1}(a)|\, f J < \frac{r}{s}\, n f J$$

and every factor of length $J\,|\zeta^{j_n}(a)|$ contains a word of the form w (as above), the result is proven with $C = rfJ/s$. ▮

Proof of Theorem 1. It follows readily from Proposition 4 above and Theorems 1 and 2 in ref. 20.

Remark. As stated in ref. 20, for such class of subshifts satisfying the linearly repetitive condition the associated hull is strictly ergodic (see ref. 19 for a proof). It then follows by this result and Proposition 4 an alternative proof of Proposition 2.

4. UNIFORM RESULTS

Notice that by Proposition 1 the substitutions in (3) are aperiodic and so, by Theorem 1, the spectra of H_ω are Cantor sets of null Lebesgue measure for all ω in the respective hulls (recall that due to strict ergodicity the spectrum, as a set, is the same for all potentials in the hull). By Proposition 3, to conclude the proof of Theorem 2, it remains to exclude eigenvalues of H_ω, for all $\omega \in \Omega_\zeta$ with ζ as in (3). This analysis is performed for each case and based on the properties of the particular potential considered. The main tools to exclude eigenvalues uniformly are Gordon type arguments combined with the concept of partitions.

In order to get uniform results, for the three subclasses $\zeta_1, \zeta_2, \zeta_3$, it will be important to establish some links (i.e., morphic images; see ahead) to the Period Doubling substitution considered in ref. 5. On initial considerations such links should suffice to get uniform results, but we decide to present a detailed analysis for two reasons: first, such links involve a uniform bound for traces maps that in principle does not extend trivially to morphic images (here it clearly takes advantage of the particular form of the Period Doubling trace map); second, these links were explicitly needed only in a special instance of the partition construction, for which we were not able to apply Proposition 5(b), and such instance have also been the only obstacle we have gotten while trying to get uniform results for still other cases of the ζ-substitution, and we want to make this occurrence clear for the interested reader (hopefully, someone could use it to obtain uniform results for all substitutions in the ζ-class).

We recall that given $\omega \in \Omega$ (in an arbitrary "hull") and $E \in \mathbb{R}$, we can construct a solution of the formal difference equation

$$(H_\omega \psi)_n = \psi_{n+1} + \psi_{n-1} + \omega_n \psi_n = E\psi_n \tag{4}$$

by using the transfer matrix formalism

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} E-\omega_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

and hence

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \underbrace{\begin{pmatrix} E-\omega_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E-\omega_1 & -1 \\ 1 & 0 \end{pmatrix}}_{M_E(\omega, n)} \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix},$$

where $M_E(\omega, n)$ is the transfer matrix from zero to n.

In the next proposition we state the two Gordon type arguments cited above (for proofs, see refs. 8 and 24):

Proposition 5. For fixed $\omega \in \Omega$ and $E \in \mathbb{R}$, let $\psi \neq 0$ be a solution of (4).

(a) If there exists a sequence $k_n \to \infty$ such that $\omega_j = \omega_{j+k_n}$, for all $1 \leqslant j \leqslant k_n$, and $0 < C_E < \infty$ satisfying $|\mathrm{tr}\, M_E(\omega, k_n)| \leqslant C_E$ ($\mathrm{tr}\, M_E$ denotes the trace of the matrix M_E), $\forall k_n$, then $\psi \notin l^2(\mathbb{Z})$, and E is not an eigenvalue of H_ω.

(b) If there exists a sequence $n_k \to \infty$ such that

$$\omega_{j-n_k} = \omega_j = \omega_{j+n_k},$$

for all $1 \leqslant j \leqslant n_k$, then $\psi \notin l^2(\mathbb{Z})$, and E is not an eigenvalue of H_ω.

An n-partition of an element ω in the hull of a bilateral ζ substitution sequence η is a decomposition of ω in substitution blocks of the form $s_n = \zeta^n(a)$ and $b = \zeta^n(b)$. For earlier use of partitions in related contexts see refs. 14 and 15; more recent applications can be found in refs. 5–7. We use the symbols $\hat{s}_n$ or $\hat{b}$ to denote the block of the n-partition that contains the zero index term of ω, and we call it the zero-block. A B_j-block in an n-partition will be called *isolated* for the $(n+1)$-partition if it remains a partition block for the $(n+1)$-partition. For example, the **b** below is isolated for the 1-partition associated to ζ_2 in (3) (here $B_1 = b$)

$$\ldots \underbrace{a\,a\,b\,a\,a}_{s_1}\,\underbrace{a\,a\,b\,a\,a}_{s_1}\;\mathbf{b}\;\underbrace{a\,a\,b\,a\,a}_{s_1}\,\underbrace{a\,a\,b\,a\,a}_{s_1}\ldots.$$

Proposition 6. Given a substitution ζ as in (3), for each $\omega \in \Omega_\zeta$ there is a unique n-partition for all integer $n \geqslant 0$.

Proof. First we discuss existence, and afterwards the uniqueness. Since a bilateral substitution sequence generated by a substitution rule in the ζ-class satisfies $\zeta^n(\eta) = \eta$ (because they begins and ends with a), it is clear the existence of an n-partition for η, the same occurring for all translations $T^j\eta$ of η. Due to the metric d defined in $\mathscr{A}^{\mathbb{Z}}$, for $\omega = \lim_{j\to\infty} T^{n_j}\eta$, for each $N \in \mathbb{N}$ there exists n_k such that for all $n_j \geqslant n_k$ we have $\omega_l = \eta_{n_j+l}$, $|l| \leqslant N$, i.e., ω and $T^{n_j}\eta$ coincide in the interval $\{-N,..., N\} \subset \mathbb{Z}$; thus, existence of the n-partition for $T^{n_j}\eta$ for all j implies existence of n-partition for all $\omega \in \Omega_\zeta$.

The uniqueness for the 0-partition is clear for each $\omega \in \Omega_\zeta$ (the proper ω). Close inspection shows that for each substitution in (3) the positions of the s_n in the n-partition uniquely determine the positions of the s_{n+1} blocks in the $(n+1)$-partition of ω, and uniqueness follows by induction. ∎

Just for sake of clarity, for ζ_1, ζ_2, and ζ_3 in (3), we restrict the details of the proof of Theorem 2 to the specific cases

$$\zeta_1(a) = abaaba, \qquad \zeta_2(a) = aabaa, \qquad \zeta_3(a) = ababbaba,$$

respectively. In each case we fix $\omega \in \Omega_\zeta$, $E \in \mathbb{R}$ and let ψ be a nonzero solution of Eq. (4). Then for an n-partition we discuss all possibilities for the zero-block position and analyze the local symmetries of the potential around it. If at least one of the possible cases of block repetitions in Proposition 5, i.e., two-blocks with uniform bound traces in (a) and three-blocks repetitions in (b), occurs for infinitely many partitions, we conclude that such a solution is not square-summable. Below we will indicate, for each possibility, if either (a) or (b) in Proposition 5 is applicable.

4.1. The ζ_1 Substitution

We begin with some simple and important properties of the elements in Ω_{ζ_1} (extended to the blocks s_n and b of the partitions):

- the "a-blocks" are either a or aa;
- in the factor bab, one of the b's is isolated for the 1-partition.

Case 1. The zero-block for the n-partition is a b.

Case 1.1. The b-block is isolated for the $(n+1)$-partition. Expanding the blocks around it, as indicated below, one sees the presence of a three-block repetitions. If this occurs for infinitely many partitions then Proposition 5(b) (reflected at the origin) can be applied

$$\ldots \overbrace{s_n\, b\, s_n\, \underbrace{s_n\, b}\, s_n}^{s_{n+1}} \hat{\mathbf{b}} \cdot \overbrace{\underbrace{s_n\, b}\, s_n\, s_n\, b\, s_n}^{s_{n+1}} \ldots .$$

Case 1.2. The b-block is not isolated. In this case it may occur:

Case 1.2.1. $s_n\,\mathbf{b}\,s_n\,\hat{b}\,s_n\,s_n\,b\,s_n$. Again by using Proposition 5(b) (reflected at the origin):

$$\ldots \overbrace{s_n\, b\, s_n\, s_n\, \underbrace{b\, s_n}}^{s_{n+1}}\, \mathbf{b}\, \overbrace{s_n\, \underbrace{\hat{b} \cdot s_n}\, s_n\, b\, s_n}^{\hat{s}_{n+1}} \ldots .$$

Analogously for $s_n\,b\,s_n\,s_n\,\hat{b}\,s_n\,\mathbf{b}\,s_n$.

Case 1.2.2. $s_n\,s_n\,\hat{b}\,s_n\,s_n$. As indicated, we can use Proposition 5(b)

$$\ldots \overbrace{s_n\, \underbrace{b\, s_n\, s_n}\, \hat{b} \cdot s_n}^{\hat{s}_{n+1}} \overbrace{s_n\, \underbrace{b\, s_n\, s_n}\, b\, s_n}^{s_{n+1}} \ldots .$$

Similarly for $s_n\,b\,s_n\,s_n\,b\,s_n\,s_n\,\hat{b} \cdot s_n\,s_n b\,s_n$.

Case 2. The zero-block is a s_n. By the structure of η we have the following possibilities:

Case 2.1. $b\hat{s}_n b$. In this case the left or the right b is isolated for the $(n+1)$-partition. Suppose that is the left b. Thus,

$$\dots \overbrace{s_n b s_n \underbrace{s_n} b s_n}^{s_{n+1}} \mathbf{b} \overbrace{\hat{s}_n b s_n s_n b s_n}^{\hat{s}_{n+1}} \dots$$

and we can apply Proposition 5(b).

Case 2.2. $b\hat{s}_n s_n b$ (analogously for $b s_n \hat{s}_n b$). We subdivide this case in the following subcases (depending on the potential structure around $\hat{s}_n$):

Case 2.2.1. $b\hat{s}_{n+1} b$, that is the Case 2.1 above.

Case 2.2.2. $\hat{s}_{n+1} s_{n+1}$ and $\hat{s}_n$ is a final block of s_{n+1},

$$\dots \overbrace{s_n b s_n \underbrace{s_n b \hat{s}_n}}^{\hat{s}_{n+1}} \overbrace{\underbrace{s_n b s_n} \underbrace{s_n b s_n}}^{s_{n+1}} \dots$$

and we apply Proposition 5(b) as indicated.

Case 2.2.3. $\hat{s}_{n+1} s_{n+1}$ and $\hat{s}_n$ is not a final block for s_{n+1} (similarly for $s_{n+1}\hat{s}_{n+1}$),

$$\dots \overbrace{s_n b s_n s_n b s_n}^{s_{n+1}} \mathbf{b} \overbrace{s_n b \hat{s}_n s_n b s_n}^{\hat{s}_{n+1}} \overbrace{s_n b s_n s_n b s_n}^{s_{n+1}} \dots .$$

In this case, due to the lack of symmetry around $\hat{s}_n$, we cannot apply Proposition 5(b). An alternative is to find explicitly bounds for the trace map, but this was not possible in this case. The way that we found to get round this difficulty was an identification between ζ_1 and the Period Doubling ξ_{pd} substitution ($\alpha \mapsto \alpha\beta$; $\beta \mapsto \alpha\alpha$):

Lemma 1. Defining $\mathcal{J}(\alpha) = abaabab$, $\mathcal{J}(\beta) = abaaba$, extended in the natural way to $\{\alpha, \beta\}^*$ and $\{\alpha, \beta\}^{\mathbb{Z}}$, then the following relations hold

$$\mathcal{J}(\xi_{\mathrm{pd}}^{2n-1}(\alpha)) = \zeta_1^n(a)\, b\zeta_1^n(a) \qquad \text{and} \qquad \mathcal{J}(\xi_{\mathrm{pd}}^{2n-1}(\beta)) = \zeta_1^n(a)\, b\zeta_1^n(a)\, b;$$
$$\mathcal{J}(\xi_{\mathrm{pd}}^{2(n-1)}(\alpha)) = \zeta_1^n(a)\, b \qquad \text{and} \qquad \mathcal{J}(\xi_{\mathrm{pd}}^{2(n-1)}(\beta)) = \zeta_1^n(a).$$

Proof. An induction argument. ▌

By using Lemma 1, we can decompose η (and hence all $\omega \in \Omega_{\zeta_1}$) in suitable Period Doubling substitution blocks:

$$\dots \underbrace{abaabab}_{\alpha}\,\underbrace{abaaba}_{\beta}\,\underbrace{abaabab}_{\alpha}\,\underbrace{abaaba}_{\beta} \cdot \underbrace{abaabab}_{\alpha}\,\underbrace{abaaba}_{\beta}\,\underbrace{abaabab}_{\alpha}\,\underbrace{abaabab}_{\alpha}\,\underbrace{abaabab}_{\alpha} \dots .$$

I.e., it is a way of "walking" over η, which will provide the necessary uniform trace bounds to apply Proposition 5(a). Define

$$x_n(E) = \mathrm{tr}(M_E(\mathcal{J}(\xi^n_{\mathrm{pd}}(\alpha)))), \qquad y_n(E) = \mathrm{tr}(M_E(\mathcal{J}(\xi^n_{\mathrm{pd}}(\beta)))).$$

Here, the transfer matrix $M_E(v)$ of a finite word v is defined in the usual way. For $n \in \mathbb{N}$, let the periodic η_n be given by

$$\eta_n = \ldots\mathcal{J}(\xi^n_{\mathrm{pd}}(\alpha)) \cdot \mathcal{J}(\xi^n_{\mathrm{pd}}(\alpha))\ldots . \tag{5}$$

By Lemma 1 and a direct calculation, each η_n contains a square $\zeta^s(a)\,\zeta^s(a)$ with $s \geqslant n/2-1$. Thus, there exists a sequence $(k(n))$ in $\mathbb{N}$ such that the operators $H_n = H_{T^{k(n)}\eta_n}$ converge in the strong sense to H_η. By standard arguments this implies

$$\sigma(H_\eta) \subset \overline{\bigcup_{k \geqslant n} \sigma_k}$$

for every $n \in \mathbb{N}$, where the bar denotes the closure in $\mathbb{R}$ and σ_k is the spectrum of H_k. As H_k is periodic, $\sigma_k = \{E\colon |x_k(E)| \leqslant 2\}$, and thus

$$\sigma(H_\eta)^c \supset \mathrm{int}\left(\bigcap_{k \geqslant n} \{E\colon |x_k(E)| > 2\}\right)$$

for every $n \in \mathbb{N}$, where int S denotes the interior of $S \subset \mathbb{R}$ and the S^c denotes the complement of S. By definition of $\mathcal{J}$ and x_k, y_k, the recursion relations (1.9) of ref. 2 hold for x_n, y_n.

These relations and the proof of Lemma 1 in ref. 2 show that $|x_k(E)| > 2$ for all $k \geqslant n$ whenever $|x_n(E)| > 2$ and $|x_{n+1}(E)| > 2$. Thus, the set

$$\bigcap_{k \geqslant n} \{E\colon |x_k(E)| > 2\} = \{E\colon |x_n(E)| > 2\} \cap \{E\colon |x_{n+1}(E)| > 2\}$$

is open (as $E \mapsto x_k(E)$ is continuous). Putting this together, we arrive at

$$\sigma(H_\eta)^c \supset \{E\colon |x_n(E)| > 2\} \cap \{E\colon |x_{n+1}(E)| > 2\}$$

for every $n \in \mathbb{N}$. This shows that

$$|x_n(E)| \leqslant 2 \qquad \text{or} \qquad |x_{n+1}(E)| \leqslant 2$$

holds for every $E \in \sigma(H_\eta)$ and $n \in \mathbb{N}$.

Now, in Case 2.2.3, adjacent to zero to the right, there appears a square vv with v being a cyclic permutation of $s_n b s_n = \mathcal{J}(\xi^{2n-1}_{\mathrm{pd}}(\alpha))$.

Adjacent to zero to the left, there appears a square ww of a cyclic permutation w of $s_n b = \mathcal{J}(\xi_{\mathrm{pd}}^{2(n-1)}(\alpha))$ (see the sketched partitions at the beginning of the discussion on Case 2.2.3). By the above reasoning at least one of the corresponding traces $x_{2(n-1)}(E)$ and $x_{2n-1}(E)$ is bounded in modulus by 2 and we are in the situation of Proposition 5(a) (either to the left or to the right).

4.2. The ζ_2 Substitution

The elements of Ω_{ζ_2} satisfies the properties (that can be extended to the blocks s_n and b):

- a always appears in the form aa or $aaaa$;
- in the factor $baab$, one of the b's is isolated for the 1-partition.

Case 1. The zero-block for the n-partition is a b-block.

We subdivide this case in the following two:

Case 1.1. b is isolated for the $(n+1)$-partition, i.e., the zero-block of the $(n+1)$-partition is a b-block. In this case, passing to the $(n+1)$-partition

$$\ldots \overbrace{\underbrace{s_n s_n}\, b\, \underbrace{s_n s_n}}^{s_{n+1}}\, \hat{\mathbf{b}} \cdot \overbrace{\underbrace{s_n s_n}\, b\, s_n s_n}^{s_{n+1}} \ldots$$

and we can apply Proposition 5(b) (reflected at the origin).

Case 1.2. b is not isolated for the $(n+1)$-partition, i.e., the zero-block of the $(n+1)$-partition is a s_{n+1}-block. Now we have the following subcases:

Case 1.2.1. $s_{n+1}\hat{s}_{n+1} b$ (or $b\hat{s}_{n+1} s_{n+1}$) and we apply Proposition 5(b) as indicated

$$\ldots \overbrace{s_n s_n b s_n s_n}^{s_{n+1}}\, \overbrace{\underbrace{s_n s_n}\, \hat{b} \cdot \underbrace{s_n s_n}}^{\hat{s}_{n+1}}\, \mathbf{b}\, \overbrace{\underbrace{s_n s_n}\, b s_n s_n}^{s_{n+1}} \ldots .$$

Case 1.2.2. $s_{n+1}\hat{s}_{n+1} s_{n+1} s_{n+1}$ (or $s_{n+1} s_{n+1} \hat{s}_{n+1} s_{n+1}$) and we apply Proposition 5(b) as indicated

$$\ldots \overbrace{s_n s_n b s_n s_n}^{s_{n+1}}\, \overbrace{s_n s_n \hat{b} \cdot s_n s_n}^{\hat{s}_{n+1}}\, \overbrace{s_n s_n b s_n s_n}^{s_{n+1}}\, \overbrace{s_n s_n b s_n s_n}^{s_{n+1}} \ldots .$$

Case 2. The zero-block to the n-partition is s_n.

We divide this case in the following:

Case 2.1. $s_n \hat{s}_n s_n$. Analogous to the Case 1.2.2.

Case 2.2. $b \hat{s}_n s_n b$. One of the b's is isolated for the $(n+1)$-partition. Suppose that is the left b. Passing to the $(n+1)$-partition

$$\cdots \mathrlap{\overbrace{}^{s_{n+1}}}\underbrace{s_n s_n b}\underbrace{s_n s_n \mathbf{b}}\,\mathrlap{\overbrace{\phantom{\hat{s}_n s_n b s_n s_n}}^{\hat{s}_{n+1}}}\underbrace{\hat{s}_n s_n b} s_n s_n \cdots$$

and we are in the conditions of Proposition 5(b) (reflected at the origin). The argument is symmetric to the isolated b position. The case $b s_n \hat{s}_n b$ is similar.

Case 2.3. $b \hat{s}_n s_n s_n s_n b$. Passing to the $(n+1)$-partition we obtain $\hat{s}_{n+1} s_{n+1}$ and have the following cases to analyze:

Case 2.3.1. $s_{n+1} \hat{s}_{n+1} s_{n+1}$, that is the Case 2.1 above.

Case 2.3.2. $b \hat{s}_{n+1} s_{n+1} b$, that is the Case 2.2.

Case 2.3.3. $b \hat{s}_{n+1} s_{n+1} s_{n+1} s_{n+1} b$, that is again the Case 2.3 that we are dealing with. We shall use the following decomposition:

$$\cdots \mathrlap{\overbrace{}^{s_{n+1}}} s_n s_n b \underbrace{s_n s_n \mathbf{b}}\,\overbrace{\underbrace{s_n s_n b}\underbrace{\hat{s}_n}\underbrace{s_n}}^{\hat{s}_{n+1}}\overbrace{s_n s_n b s_n s_n}^{s_{n+1}} \cdots .$$

If it happens for all n-partition starting from a n_0, we cannot use Proposition 5(b). To conclude this case we identify ζ_2 with the Period Doubling ξ_{pd} as in the following lemma:

Lemma 2. Defining $\mathscr{I}(\alpha) = aab$, $\mathscr{I}(\beta) = aa$, extended in the natural way to $\{\alpha, \beta\}^*$ and $\{\alpha, \beta\}^{\mathbb{Z}}$, then the following relations hold

$$\mathscr{I}(\xi_{\mathrm{pd}}^{2n-1}(\alpha)) = \zeta_2^n(a) \quad \text{and} \quad \mathscr{I}(\xi_{\mathrm{pd}}^{2n-1}(\beta)) = \zeta_2^n(a)\, b;$$
$$\mathscr{I}(\xi_{\mathrm{pd}}^{2n}(\alpha)) = \zeta_2^n(a)\, \zeta_2^n(a)\, b \quad \text{and} \quad \mathscr{I}(\xi_{\mathrm{pd}}^{2n}(\beta)) = \zeta_2^n(a)\, \zeta_2^n(a).$$

By Lemma 2 and a direct calculation, each periodic approximation of η (as in (5)) contains a square $\zeta^s(a)\, \zeta^s(a)$ with $s \geqslant n/2 - 1$. We conclude the proof as in the Case 2.2.3 for ζ_1, with a square adjacent to the right of zero of a cyclic permutation of $s_n = \mathscr{I}(\xi_{\mathrm{pd}}^{2n-1}(\alpha))$ and to the left of zero a square of a cyclic permutation of $s_n s_n b = \mathscr{I}(\xi_{\mathrm{pd}}^{2n}(\alpha))$.

4.3. The ζ_3 Substitution

For the substitution ζ_3, η_n must be defined by (see, for instance, Eq. (2))

$$\eta_n = \ldots \zeta_3^n(a)\, b \cdot \zeta_3^n(a)\, b \ldots .$$

The elements of Ω_{ζ_3} satisfy (as well as all s_n and b blocks for the correspondent partition):

- in the factor *ababa* one of *b*'s is isolated for the 1-partition;
- in the factor *bbab* just *bb* can be isolated for the 1-partition.

Case 1. The zero-block to the n-partition is s_n.

We divide this case in the following:

Case 1.1. $s_n b \hat{s}_n b s_n$. If the right b is isolated for the $(n+1)$-partition, then

$$\ldots \overbrace{s_n b s_n b b s_n b \hat{s}_n}^{\hat{s}_{n+1}} \mathbf{b} \overbrace{s_n b s_n b b s_n b s_n}^{s_{n+1}} \ldots$$

and we can apply Proposition 5(b) as indicated (analogous if the left b is isolated).

Case 1.2. $b b \hat{s}_n b$, with bb isolated for the $(n+1)$-partition. In this case

$$\ldots \overbrace{s_n b s_n b b s_n b s_n}^{s_{n+1}} \mathbf{b}\,\mathbf{b} \overbrace{\hat{s}_n b s_n b b s_n b s_n}^{\hat{s}_{n+1}} \ldots$$

and we use Proposition 5(b) as indicated. The case $b s_n b b$ with bb isolated for the $(n+1)$-partition is analogous.

Case 1.3. $b b \hat{s}_n b$, with bb not isolated for the $(n+1)$-partition. In this case, passing to the $(n+1)$-partition it may occur:

Case 1.3.1. $s_{n+1} b \hat{s}_{n+1} b s_{n+1}$ that is the Case 1.1.

Case 1.3.2. $b \hat{s}_{n+1} b b$ and we can use Proposition 5(b) as indicated

$$\ldots \overbrace{s_n b s_n b b \hat{s}_n b s_n}^{s_{n+1}} \mathbf{b}\,\mathbf{b} \overbrace{s_n b s_n b b s_n b s_n}^{\hat{s}_{n+1}} \ldots .$$

Case 1.3.3. $bb\hat{s}_{n+1}b$ that is the proper Case 1.3. We shall use the following decomposition:

$$\ldots s_n b s_n b b \underbrace{s_n b s_n \mathbf{b}\,\mathbf{b}}\, \overbrace{\underbrace{s_n b s_n b b}\underbrace{\hat{s}_n b}\, s_n}^{\hat{s}_{n+1}} \mathbf{b}\, \overbrace{s_n b s_n b b s_n b s_n}^{s_{n+1}} \ldots .$$

If it occurs for all n-partition from some n_0, we cannot use Proposition 5(b). In this case we decompose ζ_3 in Period Doubling potential blocks as follows:

Lemma 3. Defining $\mathscr{J}(\alpha) = ababb$, $\mathscr{J}(\beta) = abab$, extended in the natural way to $\{\alpha, \beta\}^*$ and $\{\alpha, \beta\}^{\mathbb{Z}}$, then the following relations hold

$$\mathscr{J}(\xi_{\mathrm{pd}}^{2n-1}(\alpha)) = \zeta_3^n(a)\, b \qquad \text{and} \qquad \mathscr{J}(\xi_{\mathrm{pd}}^{2n-1}(\beta)) = \zeta_3^n(a)\, bb;$$

$$\mathscr{J}(\xi_{\mathrm{pd}}^{2n}(\alpha)) = \zeta_3^n(a)\, b\zeta_3^n(a)\, bb \qquad \text{and} \qquad \mathscr{J}(\xi_{\mathrm{pd}}^{2n}(\beta)) = \zeta_3^n(a)\, b\zeta_3^n(a)\, b.$$

By Lemma 3 and a direct calculation, each periodic approximation of η (as in (5)) contains a square $\zeta^s(a)\, b\zeta^s(a)\, b$ with $s \geqslant n/2 - 1$. We conclude the proof as in the Case 2.2.3 for ζ_1, with a square adjacent to the right of zero of a cyclic permutation of $s_n b = \mathscr{J}(\xi_{\mathrm{pd}}^{2n-1}(\alpha))$, and to the left of zero a square of a cyclic permutation of $s_n b s_n b b = \mathscr{J}(\xi_{\mathrm{pd}}^{2n}(\alpha))$.

Case 2. The zero-block is a b-block.

Case 2.1. $s_n \hat{b} s_n$ where b is isolated. We can use Proposition 5(b) as indicated

$$\ldots s_n b s_n b b s_n b \underbrace{s_n \hat{\mathbf{b}}} \cdot \overbrace{\underbrace{s_n b}\underbrace{s_n b} b s_n b s_n}^{s_{n+1}} \ldots .$$

Case 2.2. $s_n \hat{b} b s_n$ where bb is isolated (analogous for $s_n b\hat{b} s_n$). Proposition 5(b) can be applied as follows

$$\ldots s_n b s_n b \underbrace{b s_n b s_n \hat{\mathbf{b}}} \cdot \underbrace{\mathbf{b}\, s_n b s_n b}\underbrace{b s_n b s_n b} \ldots .$$

Case 2.3. $s_n \hat{b} s_n$ where b is not isolated or the cases $s_n b\hat{b} s_n$ and $s_n \hat{b} b s_n$ with bb not isolated, reverts to Case 1.

This concludes the analysis for the ζ_3 substitution and the proof of Theorem 2.

ACKNOWLEDGMENTS

CRdeO was partially supported by CNPq (Brazil). We are grateful to Daniel Lenz for indicating mistakes and also suggesting complete corrections.

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